A simple proof that super-consistency implies cut elimination

Gilles Dowek, Olivier Hermant

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Gilles Dowek[1] and Olivier Hermant[2]

[1] École polytechnique and INRIA
LIX, École polytechnique, 91128 Palaiseau Cedex, France
gilles.dowek@polytechnique.edu
http://www.lix.polytechnique.fr/~dowek
[2] PPS, Université Denis Diderot
2 place Jussieu, 75251 Paris Cedex 05, France
hermant@pps.jussieu.fr
http://www.pps.jussieu.fr/~hermant

Abstract. We give a simple and direct proof that super-consistency implies cut elimination in deduction modulo. This proof can be seen as a simplification of the proof that super-consistency implies proof normalization. It also takes ideas from the semantic proofs of cut elimination that proceed by proving the completeness of the cut free calculus. In particular, it gives a generalization, to all super-consistent theories, of the notion of V-complex, introduced in the semantic cut elimination proofs for simple type theory.

1 Introduction

Deduction modulo is an extension of predicate logic where some axioms may be replaced by rewrite rules. For instance, the axiom $x + 0 = x$ may be replaced by the rewrite rule $x + 0 \longrightarrow x$ and the axiom $x \subseteq y \Leftrightarrow \forall z\ (z \in x \Rightarrow z \in y)$ by the rewrite rule $x \subseteq y \longrightarrow \forall z\ (z \in x \Rightarrow z \in y)$.

In the model theory of deduction modulo, it is important to distinguish the fact that some propositions are computationally equivalent, *i.e.* congruent (*e.g.* $x \subseteq y$ and $\forall z\ (z \in x \Rightarrow z \in y)$), in which case they should have the same value in a model, from the fact that they are provably equivalent, in which case they may have different values. This has lead, in [4], to introduce a generalization of Heyting algebras called *truth values algebras* and a notion of $\mathcal{B}$-valued model, where $\mathcal{B}$ is a truth values algebra. We have called *super-consistent* the theories that have a $\mathcal{B}$-valued model for all truth values algebras $\mathcal{B}$ and we have given examples of consistent theories that are not super-consistent.

In deduction modulo, there are theories for which there exists proofs that do not normalize. But, we have proved in [4] that all proofs normalize in all super-consistent theories. This proof proceeds by observing that reducibility candidates [8] can be structured in a truth values algebra and thus that super-consistent theories have reducibility candidate valued models. Then, the existence of such a model implies proof normalization [7] and hence cut elimination. As many theories, in particular arithmetic and simple type theory, are super-consistent, we get Gentzen's and Girard's theorems as corollaries.

This paper is an attempt to simplify this proof replacing the algebra of reducibility candidates $\mathcal{C}$ by a simpler truth values algebra $\mathcal{S}$. Reducibility candidates are sets of proofs. We show that we can replace each proof of such a set by its conclusion, obtaining this way sets of sequents, rather than sets of proofs, for truth values.

Although the truth values of our model are sets of sequents, our cut elimination proof uses another truth values algebra whose elements are sets of contexts: the algebra of contexts Ω, that happens to be a Heyting algebra. Besides an $\mathcal{S}$-valued model we build, for each super-consistent theory, an Ω-valued model verifying some properties, but this requires to enlarge the domain of the model using a technique of *hybridization*. The elements of such an hybrid model are quite similar to the V-complexes used in the semantic proofs of cut elimination for simple type theory [12, 13, 1, 3, 9]. Thus, we show that these proofs can be simplified using an alternative notion of V-complex and also that the V-complexes introduced for proving cut elimination of simple type theory can be used for other theories as well. This hybridization technique gives a proof that uses ideas taken from both methods used to prove cut elimination: normalization and completeness of the cut free calculus. From the first, come the ideas of truth values algebra and neutral proofs and from the second the idea of building a model such that sequents valid in this model have cut free proofs.

2 Super-consistency

To keep the paper self contained, we recall in this section the definition of deduction modulo, truth values algebras, $\mathcal{B}$-valued models and super-consistency. A more detailed presentation can be found in [4].

2.1 Deduction modulo

Deduction modulo [6, 7] is an extension of predicate logic (either single-sorted or many-sorted predicate logic) where a theory is defined by a set of axioms Γ and a congruence $\equiv$, itself defined by a confluent rewrite system rewriting terms to terms and atomic propositions to propositions.

In this paper we consider natural deduction rules. These rules are modified to take the congruence $\equiv$ into account. For example, the elimination rule of the implication is not formulated as usual

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$$

but as

$$\frac{\Gamma \vdash C \quad \Gamma \vdash A}{\Gamma \vdash B}\; C \equiv A \Rightarrow B$$

All the deduction rules are modified in a similar way, see, for instance, [7] for a complete presentation.

In deduction modulo, there are theories for which there exists proofs that do not normalize. For instance, in the theory formed with the rewrite rule $P \longrightarrow (P \Rightarrow Q)$, the proposition Q has a proof

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash P \Rightarrow Q}\text{axiom} \quad \dfrac{}{P \vdash P}\text{axiom}}{P \vdash Q}\Rightarrow\text{-elim}}{\vdash P \Rightarrow Q}\Rightarrow\text{-intro} \quad \dfrac{\dfrac{\dfrac{}{P \vdash P \Rightarrow Q}\text{axiom} \quad \dfrac{}{P \vdash P}\text{axiom}}{P \vdash Q}\Rightarrow\text{-elim}}{\vdash P}\Rightarrow\text{-intro}}{\vdash Q}\Rightarrow\text{-elim}$$

that does not normalize. In some other theories, such as the theory formed with the rewrite rule $P \longrightarrow (Q \Rightarrow P)$, all proofs strongly normalize.

In deduction modulo, like in predicate logic, closed normal proofs always end with an introduction rule. Thus, if a theory can be expressed in deduction modulo with rewrite rules only, *i.e.* with no axioms, in such a way that proofs modulo these rewrite rules strongly normalize, then the theory is consistent, it has the disjunction property and the witness property, and various proof search methods for this theory are complete.

Many theories can be expressed in deduction modulo with rewrite rules only, in particular arithmetic and simple type theory, and the notion of cut of deduction modulo subsumes the notions of cut defined for each of these theories. For instance, simple type theory can be defined as follows.

Definition 1 (Simple type theory [2, 5, 3]). *The sorts are inductively defined by*

- ι *and* o *are sorts,*
- *if* T *and* U *are sorts then* $T \to U$ *is a sort.*

The language contains the constants $S_{T,U,V}$ *of sort* $(T \to U \to V) \to (T \to U) \to T \to V$, $K_{T,U}$ *of sort* $T \to U \to T$, $\dot{\top}$ *of sort* o *and* $\dot{\bot}$ *of sort* o, $\dot{\Rightarrow}$, $\dot{\wedge}$ *and* $\dot{\vee}$ *of sort* $o \to o \to o$, $\dot{\forall}_T$ *and* $\dot{\exists}_T$ *of sort* $(T \to o) \to o$, *the function symbols* $\alpha_{T,U}$ *of rank* $\langle T \to U, T, U\rangle$ *and the predicate symbol* ε *of rank* $\langle o \rangle$.

The rules are

$$\alpha(\alpha(\alpha(S_{T,U,V}, x), y), z) \longrightarrow \alpha(\alpha(x, z), \alpha(y, z))$$
$$\alpha(\alpha(K_{T,U}, x), y) \longrightarrow x$$
$$\varepsilon(\dot{\top}) \longrightarrow \top$$
$$\varepsilon(\dot{\bot}) \longrightarrow \bot$$
$$\varepsilon(\alpha(\alpha(\dot{\Rightarrow}, x), y)) \longrightarrow \varepsilon(x) \Rightarrow \varepsilon(y)$$
$$\varepsilon(\alpha(\alpha(\dot{\wedge}, x), y)) \longrightarrow \varepsilon(x) \wedge \varepsilon(y)$$
$$\varepsilon(\alpha(\alpha(\dot{\vee}, x), y)) \longrightarrow \varepsilon(x) \vee \varepsilon(y)$$
$$\varepsilon(\alpha(\dot{\forall}_T, x)) \longrightarrow \forall y\ \varepsilon(\alpha(x, y))$$
$$\varepsilon(\alpha(\dot{\exists}_T, x)) \longrightarrow \exists y\ \varepsilon(\alpha(x, y))$$

2.2 Truth values algebras

Definition 2 (Truth values algebra). *Let B be a set, whose elements are called* truth values*, B^+ be a subset of B, whose elements are called* positive truth values*, $\mathcal{A}$ and $\mathcal{E}$ be subsets of $\wp(B)$, $\tilde{\top}$ and $\tilde{\bot}$ be elements of B, $\tilde{\Rightarrow}$, $\tilde{\wedge}$, and $\tilde{\vee}$ be functions from $B \times B$ to B, $\tilde{\forall}$ be a function from $\mathcal{A}$ to B and $\tilde{\exists}$ be a function from $\mathcal{E}$ to B. The structure $\mathcal{B} = \langle B, B^+, \mathcal{A}, \mathcal{E}, \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists}\rangle$ is said to be* a truth values algebra *if the set B^+ is* closed by the intuitionistic deduction rules i.e. *if for all a, b, c in B, A in $\mathcal{A}$ and E in $\mathcal{E}$,*

1. *if $a \tilde{\Rightarrow} b \in B^+$ and $a \in B^+$ then $b \in B^+$,*
2. *$a \tilde{\Rightarrow} b \tilde{\Rightarrow} a \in B^+$,*
3. *$(a \tilde{\Rightarrow} b \tilde{\Rightarrow} c) \tilde{\Rightarrow} (a \tilde{\Rightarrow} b) \tilde{\Rightarrow} a \tilde{\Rightarrow} c \in B^+$,*
4. *$\tilde{\top} \in B^+$,*
5. *$\tilde{\bot} \tilde{\Rightarrow} a \in B^+$,*
6. *$a \tilde{\Rightarrow} b \tilde{\Rightarrow} (a \tilde{\wedge} b) \in B^+$,*
7. *$(a \tilde{\wedge} b) \tilde{\Rightarrow} a \in B^+$,*
8. *$(a \tilde{\wedge} b) \tilde{\Rightarrow} b \in B^+$,*
9. *$a \tilde{\Rightarrow} (a \tilde{\vee} b) \in B^+$,*
10. *$b \tilde{\Rightarrow} (a \tilde{\vee} b) \in B^+$,*
11. *$(a \tilde{\vee} b) \tilde{\Rightarrow} (a \tilde{\Rightarrow} c) \tilde{\Rightarrow} (b \tilde{\Rightarrow} c) \tilde{\Rightarrow} c \in B^+$,*
12. *the set $a \tilde{\Rightarrow} A = \{a \tilde{\Rightarrow} e \mid e \in A\}$ is in $\mathcal{A}$ and the set $E \tilde{\Rightarrow} a = \{e \tilde{\Rightarrow} a \mid e \in E\}$ is in $\mathcal{A}$,*
13. *if all elements of A are in B^+ then $\tilde{\forall}\ A \in B^+$,*
14. *$\tilde{\forall}\ (a \tilde{\Rightarrow} A) \tilde{\Rightarrow} a \tilde{\Rightarrow} (\tilde{\forall}\ A) \in B^+$,*
15. *if $a \in A$, then $(\tilde{\forall}\ A) \tilde{\Rightarrow} a \in B^+$,*
16. *if $a \in E$, then $a \tilde{\Rightarrow} (\tilde{\exists}\ E) \in B^+$,*
17. *$(\tilde{\exists}\ E) \tilde{\Rightarrow} \tilde{\forall}\ (E \tilde{\Rightarrow} a) \tilde{\Rightarrow} a \in B^+$.*

Proposition 1. *Any Heyting algebra is a truth values algebra. The operations $\tilde{\top}$, $\tilde{\wedge}$, $\tilde{\forall}$ are greatest lower bounds, the operations $\tilde{\bot}$, $\tilde{\vee}$, $\tilde{\exists}$ are least upper bounds, the operation $\tilde{\Rightarrow}$ is the arrow of the Heyting algebra, and $B^+ = \{\tilde{\top}\}$.*

Proof. See [4].

Definition 3 (Full). *A truth values algebra is said to be* full *if $\mathcal{A} = \mathcal{E} = \wp(B)$,* i.e. *if $\tilde{\forall}\ A$ and $\tilde{\exists}\ A$ exist for all subsets A of B.*

Definition 4 (Ordered truth values algebra). *An* ordered truth values algebra *is a truth values algebra together with a relation $\sqsubseteq$ on B such that*

- *$\sqsubseteq$ is an order relation,* i.e. *a reflexive, antisymmetric and transitive relation,*
- *B^+ is upward closed,*
- *$\tilde{\top}$ and $\tilde{\bot}$ are maximal and minimal elements,*
- *$\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\forall}$ and $\tilde{\exists}$ are monotone, $\tilde{\Rightarrow}$ is left anti-monotone and right monotone.*

Definition 5 (Complete ordered truth values algebra). *A ordered truth values algebra is said to be* complete *if every subset of B has a greatest lower bound for $\sqsubseteq$.*

2.3 Models

Definition 6 ($\mathcal{B}$-structure). *Let $\mathcal{L} = \langle f_i, P_j \rangle$ be a language in predicate logic and $\mathcal{B}$ be a truth values algebra, a $\mathcal{B}$*-structure *$\mathcal{M} = \langle M, \mathcal{B}, \hat{f}_i, \hat{P}_j \rangle$, for the language $\mathcal{L}$, is a structure such that $\hat{f}_i$ is a function from M^n to M where n is the arity of the symbol f_i and $\hat{P}_j$ is a function from M^n to B where n is the arity of the symbol P_i.*

This definition extends trivially to many-sorted languages.

Definition 7 (Denotation). *Let $\mathcal{B}$ be a truth values algebra, $\mathcal{M}$ be a $\mathcal{B}$-structure and ϕ be an assignment. The denotation $[\![A]\!]_\phi$ of a proposition A in $\mathcal{M}$ is defined as follows*

- $[\![x]\!]_\phi = \phi(x)$,
- $[\![f(t_1, ..., t_n)]\!]_\phi = \hat{f}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![P(t_1, ..., t_n)]\!]_\phi = \hat{P}([\![t_1]\!]_\phi, ..., [\![t_n]\!]_\phi)$,
- $[\![\top]\!]_\phi = \tilde{\top}$,
- $[\![\bot]\!]_\phi = \tilde{\bot}$,
- $[\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \;\tilde{\Rightarrow}\; [\![B]\!]_\phi$,
- $[\![A \wedge B]\!]_\phi = [\![A]\!]_\phi \;\tilde{\wedge}\; [\![B]\!]_\phi$,
- $[\![A \vee B]\!]_\phi = [\![A]\!]_\phi \;\tilde{\vee}\; [\![B]\!]_\phi$,
- $[\![\forall x\ A]\!]_\phi = \tilde{\forall}\ \{[\![A]\!]_{\phi+(d/x)} \mid d \in M\}$,
- $[\![\exists x\ A]\!]_\phi = \tilde{\exists}\ \{[\![A]\!]_{\phi+(d/x)} \mid d \in M\}$.

Notice that the denotation of a proposition containing quantifiers may be undefined, but it is always defined if the truth values algebra is full.

Definition 8 (Denotation of a context and of a sequent). *The denotation $[\![A_1, ..., A_n]\!]_\phi$ of a context $A_1, ..., A_n$ is that of the proposition $A_1 \wedge ... \wedge A_n$. The denotation $[\![A_1, ..., A_n \vdash B]\!]_\phi$ of the sequent $A_1, ..., A_n \vdash B$ is that of the proposition $(A_1 \wedge ... \wedge A_n) \Rightarrow B$.*

Definition 9 (Model). *A proposition A is said to be* valid *in a $\mathcal{B}$-structure $\mathcal{M}$, and the $\mathcal{B}$-structure $\mathcal{M}$ is said to be a* model of A *if for all assignments ϕ, $[\![A]\!]_\phi$ is defined and is a positive truth value.*

Consider a theory in deduction modulo defined by a set of axioms Γ and a congruence $\equiv$. The $\mathcal{B}$-structure $\mathcal{M}$ is said to be a model of the theory $\Gamma, \equiv$ if all axioms of Γ are valid in $\mathcal{M}$ and for all terms or propositions A and B such that $A \equiv B$ and assignments ϕ, $[\![A]\!]_\phi$ and $[\![B]\!]_\phi$ are defined and $[\![A]\!]_\phi = [\![B]\!]_\phi$.

Deduction modulo is sound and complete with respect to this notion of model.

Proposition 2 (Soundness and completeness). *The proposition A is provable in the theory formed with the axioms Γ and the congruence $\equiv$ if and only if it is valid in all the models of $\Gamma, \equiv$ where the truth values algebra is full, ordered and complete.*

Proof. See [4].

2.4 Super-consistency

Definition 10 (Super-consistent). *A theory in deduction modulo formed with the axioms Γ and the congruence $\equiv$ is* super-consistent *if it has a $\mathcal{B}$-valued model for all full, ordered and complete truth values algebras $\mathcal{B}$.*

Proposition 3. *Simple type theory is super-consistent.*

Proof. Let $\mathcal{B}$ be a full truth values algebra. We build the model $\mathcal{M}$ as follows. The domain M_ι is any non empty set, for instance the singleton $\{0\}$, the domain M_o is $\mathcal{B}$ and the domain $M_{T\to U}$ is the set $M_U^{M_T}$ of functions from M_T to M_U. The interpretation of the symbols of the language is $\hat{S}_{T,U,V} = a \mapsto (b \mapsto (c \mapsto a(c)(b(c))))$, $\hat{K}_{T,U} = a \mapsto (b \mapsto a)$, $\hat{\alpha}(a,b) = a(b)$, $\hat{\varepsilon}(a) = a$, $\hat{\dot{\top}} = \tilde{\top}$, $\hat{\dot{\bot}} = \tilde{\bot}$, $\hat{\dot{\Rightarrow}} = \tilde{\Rightarrow}$, $\hat{\dot{\wedge}} = \tilde{\wedge}$, $\hat{\dot{\vee}} = \tilde{\vee}$, $\hat{\dot{\forall}}_T = a \mapsto \tilde{\forall}(Range(a))$, $\hat{\dot{\exists}}_T = a \mapsto \tilde{\exists}(Range(a))$ where $Range(a)$ is the range of the function a. The model $\mathcal{M}$ is a $\mathcal{B}$-valued model of simple type theory.

3 Cut elimination

3.1 The algebra of sequents

Definition 11 (Neutral proof). *A proof is said to be* neutral *if its last rule is the axiom rule or an elimination rule, but not an introduction rule.*

Definition 12 (A positive definition of cut free proofs). Cut free *proofs are defined inductively as follows:*

- *a proof that ends with the axiom rule is cut free,*
- *a proof that ends with an introduction rule and where the premises of the last rule are proved with cut free proofs is cut free,*
- *a proof that ends with an elimination rule and where the major premise of the last rule is proved with a neutral cut free proof and the other premises with cut free proofs is cut free.*

Definition 13 (The algebra of sequents).

- *$\tilde{\top}$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut free proof or such that $C \equiv \top$.*
- *$\tilde{\bot}$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut free proof.*
- *$a \mathbin{\tilde{\wedge}} b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut free proof or such that $C \equiv (A \wedge B)$ with $(\Gamma \vdash A) \in a$ and $(\Gamma \vdash B) \in b$.*
- *$a \mathbin{\tilde{\vee}} b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut free proof or such that $C \equiv (A \vee B)$ with $(\Gamma \vdash A) \in a$ or $(\Gamma \vdash B) \in b$.*
- *$a \mathbin{\tilde{\Rightarrow}} b$ is the set of sequents $\Gamma \vdash C$ that have a neutral cut free proof or such that $C \equiv (A \Rightarrow B)$ and for all contexts Σ such that $(\Gamma, \Sigma \vdash A) \in a$, we have $(\Gamma, \Sigma \vdash B) \in b$.*

- $\tilde{\forall}\ S$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv (\forall x\ A)$ *and for every term* t *and every* a *in* S, $(\Gamma \vdash (t/x)A) \in a$.
- $\tilde{\exists}\ S$ *is the set of sequents* $\Gamma \vdash C$ *that have a neutral cut free proof or such that* $C \equiv (\exists x\ A)$ *and for some term* t *and some* a *in* S, $(\Gamma \vdash (t/x)A) \in a$.

Let S *be the smallest set of sets of sequents closed by* $\tilde{\top}$, $\tilde{\bot}$, $\tilde{\wedge}$, $\tilde{\vee}$, $\tilde{\Rightarrow}$, $\tilde{\forall}$, $\tilde{\exists}$ *and by arbitrary intersections.*

Proposition 4. *The structure* $\mathcal{S} = \langle S, S, \wp(S), \wp(S), \tilde{\top}, \tilde{\bot}, \tilde{\Rightarrow}, \tilde{\wedge}, \tilde{\vee}, \tilde{\forall}, \tilde{\exists}, \subseteq\rangle$ *is a full, ordered and complete truth values algebra.*

Proof. As all truth values are positive, the conditions of Definition 2 are obviously met. Thus $\mathcal{S}$ is a truth values algebra. As the domains of $\tilde{\forall}$ and $\tilde{\exists}$ are defined as $\wp(S)$, this algebra is full. As it is closed by arbitrary intersections, all subsets of S have a greatest lower bound, thus all subsets of S have a least upper bound and the algebra is complete.

Remark. The algebra $\mathcal{S}$ is not a Heyting algebra. In particular $\tilde{\top}\ \tilde{\wedge}\ \tilde{\top}$ and $\tilde{\top}$ are different: the first set contains the sequent $\vdash \top \wedge \top$, but not the second.

Proposition 5. *For all elements* a *of* S, *contexts* Γ, *and propositions* A *and* B

- $(\Gamma, A \vdash A) \in a$,
- *if* $(\Gamma \vdash B) \in a$ *then* $(\Gamma, A \vdash B) \in a$,
- *if* $(\Gamma \vdash A) \in a$ *and* $B \equiv A$ *then* $(\Gamma \vdash B) \in a$,
- *if* $(\Gamma \vdash A) \in a$ *then* $\Gamma \vdash A$ *has a cut free proof.*

Proof. The first proposition is proved by noticing that the sequent $\Gamma, A \vdash A$ has a neutral cut free proof. The others are proved by simple inductions on the construction of a.

Consider a super-consistent theory $\Gamma, \equiv$. As this theory is super-consistent, it has an $\mathcal{S}$-model $\mathcal{M}$. In the rest of the paper, $\mathcal{M}$ refers to this fixed model whose domain is written M.

3.2 The algebra of contexts

Definition 14 (Fiber). *Let* b *be a set of sequents and* A *a proposition, we define the fiber over* A *in* b, $b \lhd A$, *as the set of contexts* Γ *such that* $(\Gamma \vdash A) \in b$.

Definition 15 (Outer value [10, 11]). *Let* A *be a proposition,* ϕ *be an assignment and* σ *a substitution, we define the set of contexts* $[A]^\sigma_\phi$ *as the set* $[\![A]\!]_\phi \lhd \sigma A$ i.e. $\{\Gamma \mid (\Gamma \vdash \sigma A) \in [\![A]\!]_\phi\}$.

Proposition 6. *For all contexts* Γ, *propositions* A *and* B, *substitutions* σ, *and* $\mathcal{M}$*-assignments* ϕ

- $(\Gamma, \sigma A) \in [A]^\sigma_\phi$,
- *if* $\Gamma \in [B]^\sigma_\phi$ *then* $(\Gamma, A) \in [B]^\sigma_\phi$,

– *if* $\Gamma \in [A]^\sigma_\phi$ *and* $B \equiv A$ *then* $\Gamma \in [B]^\sigma_\phi$,
– *if* $\Gamma \in [A]^\sigma_\phi$ *then* $\Gamma \vdash \sigma A$ *has a cut free proof.*

Proof. From Proposition 5.

Definition 16 (The algebra of contexts). *Let* Ω *be the smallest set of sets of contexts containing all the* $[A]^\sigma_\phi$ *for some proposition* A*, assignment* ϕ*, and substitution* σ *and closed by arbitrary intersections.*

Notice that an element c of Ω can always be written in the form

$$c = \bigcap_{i \in \Lambda_c} [A_i]^{\sigma_i}_{\phi_i}$$

Proposition 7. *The set* Ω *ordered by inclusion is a complete Heyting algebra.*

Proof. As Ω is ordered by inclusion, the greatest lower bound of a subset of Ω is the intersection of all its elements. As Ω is closed by arbitrary intersections, all its subsets have greatest lower bounds. Thus, all its subsets also have least upper bounds.

The operations $\check{\top}$, $\check{\wedge}$ and $\check{\forall}$ are defined as nullary, binary and infinitary greatest lower bounds and the operations $\check{\bot}$, $\check{\vee}$ and $\check{\exists}$ are defined as nullary, binary and infinitary least upper bounds. Finally, the arrow $\check{\Rightarrow}$ of two elements a and b is the least upper bound of all the c in Ω such that $a \cap c \leq b$

$$a \check{\Rightarrow} b = \check{\exists}\ \{c \in \Omega \mid a \cap c \leq b\}$$

Notice that the nullary least upper bound $\check{\bot}$ is the intersection of all the elements of Ω that contain the empty set, *i.e.* the intersection of all the elements of Ω.

The binary least upper bound, $a \check{\vee} b$, of a and b is the intersection of all the elements of Ω that contain $a \cup b$. From Definition 16

$$a \check{\vee} b = \bigcap_{(a \cup b) \subseteq c} c = \bigcap_{(a \cup b) \subseteq \bigcap [A_i]^{\sigma_i}_{\phi_i}} (\bigcap [A_i]^{\sigma_i}_{\phi_i}) = \bigcap_{(a \cup b) \subseteq [A]^\sigma_\phi} [A]^\sigma_\phi$$

The infinitary least upper bound $\check{\exists}\, E$ of the elements of a set E is the intersection of all the elements of Ω that contain the union of the elements of E. For the same reason as above

$$\check{\exists}\ E = \bigcap_{(\bigcup E)\ \subseteq\ c} c = \bigcap_{(\bigcup E)\ \subseteq\ [A]^\sigma_\phi} [A]^\sigma_\phi$$

Finally, notice that Ω is a non trivial Heyting algebra, although the Heyting algebra $\mathcal{S}/S^+$ is trivial because $S^+ = S$.

The next proposition, the Key lemma of our proof, shows that the outer values of compound propositions can be obtained from the outer values of their components using the suitable operation of the Heyting algebra Ω. Notice that, unlike most semantic cut elimination proofs, we directly prove equalities in this lemma, and not just inclusions, although the cut elimination proof is not completed yet.

Proposition 8 (Key lemma). *For all substitutions σ, assignments ϕ and propositions A and B*

- $[\top]^\sigma_\phi = \check{\top}$,
- $[\bot]^\sigma_\phi = \check{\bot}$,
- $[A \wedge B]^\sigma_\phi = [A]^\sigma_\phi \check{\wedge} [B]^\sigma_\phi$,
- $[A \vee B]^\sigma_\phi = [A]^\sigma_\phi \check{\vee} [B]^\sigma_\phi$,
- $[A \Rightarrow B]^\sigma_\phi = [A]^\sigma_\phi \check{\Rightarrow} [B]^\sigma_\phi$,
- $[\forall x A]^\sigma_\phi = \check{\forall} \{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in M\}$,
- $[\exists x A]^\sigma_\phi = \check{\exists} \{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in M\}$.

where $\mathcal{T}$ is the set of open terms in the language of this theory.

Proof.
- By Definition 13, for any Γ, $(\Gamma \vdash \top) \in \tilde{\top}$. Thus $[\top]^\sigma_\phi = \Omega = \check{\top}$.
- The set $\check{\bot}$ is the intersection of all $[C]^\tau_\rho$. In particular, $\check{\bot} \subseteq [\bot]^\sigma_\phi$. Conversely, let $\Gamma \in [\bot]^\sigma_\phi$. Consider arbitrary C, ρ and τ. By Definition 13, $\Gamma \vdash \sigma\bot$ has a neutral cut free proof. So does $\Gamma \vdash \tau C$ and thus $\Gamma \in [C]^\tau_\rho$. Hence Γ is an element of all $[C]^\tau_\rho$ and therefore of their intersection $\check{\bot}$.
- Let $\Gamma \in [A]^\sigma_\phi \check{\wedge} [B]^\sigma_\phi = [A]^\sigma_\phi \cap [B]^\sigma_\phi$. We have $\Gamma \in [A]^\sigma_\phi$ and $\Gamma \in [B]^\sigma_\phi$ and thus $(\Gamma \vdash \sigma A) \in [\![A]\!]_\phi$ and $(\Gamma \vdash \sigma B) \in [\![B]\!]_\phi$. From Definition 13, we get $(\Gamma \vdash \sigma(A \wedge B)) \in [\![A \wedge B]\!]_\phi$. Hence $\Gamma \in [A \wedge B]^\sigma_\phi$. Conversely, let $\Gamma \in [A \wedge B]^\sigma_\phi$, we have $(\Gamma \vdash \sigma(A \wedge B)) \in ([\![A]\!]_\phi \tilde{\wedge} [\![B]\!]_\phi)$. If $\Gamma \vdash \sigma(A \wedge B)$ has a neutral and cut free proof, then so do $\Gamma \vdash \sigma A$ and $\Gamma \vdash \sigma B$. Thus $\Gamma \in [A]^\sigma_\phi$ and $\Gamma \in [B]^\sigma_\phi$, hence $\Gamma \in [A]^\sigma_\phi \cap [B]^\sigma_\phi = \Gamma \in [A]^\sigma_\phi \check{\wedge} [B]^\sigma_\phi$. Otherwise we directly have $\Gamma \in [A]^\sigma_\phi$ and $\Gamma \in [B]^\sigma_\phi$ and we conclude the same way.
- Let us first prove $[A]^\sigma_\phi \check{\vee} [B]^\sigma_\phi \subseteq [A \vee B]^\sigma_\phi$. It is sufficient to prove that $[A \vee B]^\sigma_\phi$ is an upper bound of $[A]^\sigma_\phi$ and $[B]^\sigma_\phi$. Let $\Gamma \in [A]^\sigma_\phi$. We have $(\Gamma \vdash \sigma A) \in [\![A]\!]_\phi$. By Definition 13, $(\Gamma \vdash \sigma(A \vee B)) \in ([\![A]\!]_\phi \tilde{\vee} [\![B]\!]_\phi) = [\![A \vee B]\!]_\phi$. Thus $\Gamma \in [A \vee B]^\sigma_\phi$. We prove, in a similar way, that if $\Gamma \in [B]^\sigma_\phi$ then $\Gamma \in [A \vee B]^\sigma_\phi$. Conversely, let $\Gamma \in [A \vee B]^\sigma_\phi$. Let C, ρ and τ such that $[A]^\sigma_\phi \cup [B]^\sigma_\phi \subseteq [C]^\tau_\rho$. We have $(\Gamma \vdash \sigma(A \vee B)) \in ([\![A]\!]_\phi \tilde{\vee} [\![B]\!]_\phi)$. From Definition 13, there are three cases to consider. First, if $\Gamma \vdash \sigma(A \vee B)$ has a neutral cut free proof. As $(\Gamma, \sigma A) \in [A]^\sigma_\phi \subseteq [C]^\tau_\rho$, the sequent $\Gamma, \sigma A \vdash \tau C$ has a cut free proof by Proposition 6. In a similar way, the sequent $\Gamma, \sigma B \vdash \tau C$ has a cut free proof. Hence, we can apply the $\vee$-elim rule and obtain a neutral cut free proof of $\Gamma \vdash \tau C$. Thus $\Gamma \in [C]^\tau_\rho$. As Γ is an element of all such upper bounds, it is an element of their intersection i.e. of $[A]^\sigma_\phi \check{\vee} [B]^\sigma_\phi$. Second, if $(\Gamma \vdash \sigma A) \in [\![A]\!]_\phi$. We have $\Gamma \in [A]^\sigma_\phi \subseteq [C]^\tau_\rho$. Again, Γ is an element of their intersection. The case $(\Gamma \vdash \sigma B) \in [\![B]\!]_\phi$ is similar.

- Let us prove $[A \Rightarrow B]^\sigma_\phi \subseteq [A]^\sigma_\phi \tilde{\Rightarrow} [B]^\sigma_\phi$. This is equivalent to $[A]^\sigma_\phi \cap [A \Rightarrow B]^\sigma_\phi \subseteq [B]^\sigma_\phi$. Suppose $(\Gamma \vdash \sigma A) \in [\![A]\!]_\phi$ and $(\Gamma \vdash \sigma A \Rightarrow \sigma B) \in [\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \tilde{\Rightarrow} [\![B]\!]_\phi$. If $\Gamma \vdash \sigma A \Rightarrow \sigma B$ has a neutral cut free proof, so does $\Gamma \vdash \sigma B$, as $\Gamma \vdash \sigma A$ has a cut free proof. Thus $\Gamma \in [B]^\sigma_\phi$. Otherwise, considering an empty context Σ in Definition 13, we have $(\Gamma \vdash \sigma A) \in [\![A]\!]_\phi$ and thus we get $(\Gamma \vdash \sigma B) \in [\![B]\!]_\phi$. Conversely let us prove $[A]^\sigma_\phi \tilde{\Rightarrow} [B]^\sigma_\phi \subseteq [A \Rightarrow B]^\sigma_\phi$. We have to prove that $[A \Rightarrow B]^\sigma_\phi$ is an upper bound of the set of all the $c \in \Omega$ such that $c \cap [A]^\sigma_\phi \subseteq [B]^\sigma_\phi$. Let such a c, we have to prove $c \subseteq [A \Rightarrow B]^\sigma_\phi$. As noticed c has the form $\bigcap [C_i]^{\tau_i}_{\rho_i}$. Let $\Gamma \in c$. We must show $(\Gamma \vdash \sigma A \Rightarrow \sigma B) \in [\![A \Rightarrow B]\!]_\phi = [\![A]\!]_\phi \tilde{\Rightarrow} [\![B]\!]_\phi$. For this, let Σ such that $(\Gamma, \Sigma \vdash \sigma A) \in [\![A]\!]_\phi$. This is equivalent to $\Gamma, \Sigma \in [A]^\sigma_\phi$. By Proposition 6, we know that $\Gamma, \Sigma \in [C_i]^{\tau_i}_{\rho_i}$. Therefore it is an element of their intersection. Thus $\Gamma, \Sigma \in [B]^\sigma_\phi$. Finally $(\Gamma, \Sigma \vdash \sigma B) \in [\![B]\!]_\phi$. Hence $c \subseteq [A \Rightarrow B]^\sigma_\phi$.
- Let $\Gamma \in \bigcap \{[A]^{\sigma+(t/x)}_{\phi+(d/x)}, t \in \mathcal{T}, d \in M\}$. Then we have for any t and any d, $(\Gamma \vdash (\sigma + (t/x))A) \in [\![A]\!]_{\phi+(d/x)}$. Hence, $(\Gamma \vdash \sigma \forall x A) \in \tilde{\forall}\{[\![A]\!]_{\phi+(d/x)}, d \in M\} = [\![\forall x A]\!]_\phi$. Conversely, let $\Gamma \in [\forall x A]^\sigma_\phi$. Then $(\Gamma \vdash \sigma \forall x A) \in [\![\forall x A]\!]_\phi$. If $\Gamma \vdash \sigma \forall x A$ has a neutral cut free proof then so does the sequent $\Gamma \vdash (\sigma + (t/x))A$ for any t and this sequent is an element of $[\![A]\!]_{\phi+(d/x)}$ for any d. Hence $\Gamma \in [A]^{\sigma+(t/x)}_{\phi+(d/x)}$ for any t, d. So, it is an element of their intersection. Otherwise, by Definition 13, for all t and d we have $(\Gamma \vdash (\sigma + (t/x))A) \in [\![A]\!]_{\phi+(d/x)}$ thus Γ is an element of the intersection.
- We first prove that for any t, d, $[\exists x A]^\sigma_\phi$ is an upper bound of the set $\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in M\}$. Consider some t, d and $\Gamma \in [A]^{\sigma+(t/x)}_{\phi+(d/x)}$. We have $(\Gamma \vdash (\sigma + (t/x))A) \in [\![A]\!]_{\phi+(d/x)}$. By Definition 13, $(\Gamma \vdash \sigma \exists x A) \in \tilde{\exists}\{[\![A]\!]_{\phi+(d/x)}, d \in M\}$. Hence $\Gamma \in [\exists x A]^\sigma_\phi$ and for any t and d, $[A]^{\sigma+(t/x)}_{\phi+(d/x)} \subseteq [\exists x A]^\sigma_\phi$. So $\check{\exists}\ \{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in M\} \subseteq [\exists x A]^\sigma_\phi$. Conversely, let $\Gamma \in [\exists x A]^\sigma_\phi$. Suppose $\Gamma \vdash \sigma \exists x A$ has a neutral cut free proof. Let $u = \bigcap [C_i]^{\tau_i}_{\rho_i}$ an upper bound of $\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in M\}$. We can choose $u = [C]^\tau_\rho$, since we need the intersection of the upper bounds. Let $\phi' = \phi + (d/x)$ and $\sigma' = \sigma + (y/x)$ where y is a variable appearing neither in ϕ nor in σ (the choice of $d \in M$ is immaterial). By hypothesis: $[A]^{\sigma'}_{\phi'} \subseteq [C]^\tau_\rho$. By Proposition 6, $\sigma' A \in [A]^{\sigma'}_{\phi'}$. Hence $\sigma' A \in [C]^\tau_\rho$. Thus the sequent $(\sigma' A \vdash \tau C) \in [\![C]\!]_\rho$ has a cut free proof by Proposition 6. Thus so does the sequent $\Gamma \vdash \tau C$. Hence $\Gamma \in [C]^\tau_\rho$. This is valid for any $[C]^\tau_\rho$ upper bound of $\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in M\}$. So, Γ is in their intersection, that is $\tilde{\exists}\{[A]^{\sigma+(t/x)}_{\phi+(d/x)}, t \in \mathcal{T}, d \in M\}$ Otherwise by Definition 13, $\Gamma \vdash \sigma \exists x A$ is such that for some term t and element d, $(\Gamma \vdash (\sigma + (t/x))A) \in [\![A]\!]_{\phi+(d/x)}$. This shows that $\Gamma \in [A]^{\sigma+(t/x)}_{\phi+(d/x)}$. Then $\Gamma \in \tilde{\exists}\{[A]^{\sigma+(t/x)}_{\phi+(d/x)} \mid t \in \mathcal{T}, d \in M\}$.

Proposition 9. $\sigma\Gamma \in [\Gamma]^\sigma_\phi$

Proof. Let $\Gamma = A_1, ..., A_n$. Recall that, by Definition 8 and Proposition 8, $[\Gamma]^\sigma_\phi = [A_1 \wedge ... \wedge A_n]^\sigma_\phi = [A_1]^\sigma_\phi \check{\wedge} ... \check{\wedge} [A_n]^\sigma_\phi$. Using Proposition 6, we have $\sigma\Gamma \in [A_1]^\sigma_\phi$, ..., $\sigma\Gamma \in [A_n]^\sigma_\phi$, thus $\sigma\Gamma \in ([A_1]^\sigma_\phi \check{\wedge} ... \check{\wedge} [A_n]^\sigma_\phi)$.

3.3 Hybridization and cut elimination

A usual way to prove cut elimination would be to prove by induction over proof structure that if the sequent $\Gamma \vdash B$ is provable then for every substitution σ and every valuation ϕ, $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$. Then using the fact that $\Gamma \in [\Gamma]^\varnothing_\phi$ (Proposition 9) we can prove $\Gamma \in [B]^\varnothing_\phi$ and conclude, with Proposition 6, that $\Gamma \vdash B$ has a cut free proof.

We shall follow a slightly different way here and construct another model $\mathcal{D}$, built from $\mathcal{M}$ and Ω-valued, such that the denotation of a proposition A is the value $[A]^\sigma_\phi$. Then the fact that if $\Gamma \vdash B$ is provable then $[\Gamma]^\sigma_\phi \subseteq [B]^\sigma_\phi$ will be just a consequence of the soundness theorem. The elements of the domain of the model $\mathcal{D}$ are quite similar to the V-complexes used in the proofs of cut elimination for simple type theory that proceed by proving the completeness of the cut free calculus. So we give a generalization of the notion of V-complex that can be used not only for simple type theory but also for all super-consistent theories.

Consider a super-consistent theory and its model $\mathcal{M}$ defined in Section 3.

Definition 17 (The model $\mathcal{D}$). *The model $\mathcal{D}$ is an Ω-valued model with domain $D = \mathcal{T}' \times M$ where $\mathcal{T}'$ is the set of (classes modulo $\equiv$ of) open terms of the language of the theory and M the domain of the model $\mathcal{M}$.*

Let f be a function symbol of the language and $\hat{f}^{\mathcal{M}}$ its interpretation in the model $\mathcal{M}$, the interpretation $\hat{f}^{\mathcal{D}}$ of this symbol in the model $\mathcal{D}$ is the function from D^n to D

$$\langle t_1, a_1\rangle, ..., \langle t_n, a_n\rangle \mapsto \langle f(t_1, ..., t_n), \hat{f}^{\mathcal{M}}(a_1, ..., a_n)\rangle$$

Let P be a predicate symbol of the language and $\hat{P}^{\mathcal{M}}$ its interpretation in the model $\mathcal{M}$. The interpretation $\hat{P}^{\mathcal{D}}$ of this symbol in the model $\mathcal{D}$ is the function from D^n to Ω

$$\langle t_1, a_1\rangle, ..., \langle t_n, a_n\rangle \mapsto \hat{P}^{\mathcal{M}}(a_1, ..., a_n) \lhd P(t_1, ..., t_n)$$

Let ψ be an assignment mapping variables to elements $\langle t, d\rangle$ of D. We write ψ^1 for the substitution mapping the variable x to a fixed representative of the first component of ψx and ψ^2 for the $\mathcal{M}$-assignment mapping x to the second component of ψx. Notice that, by Proposition 6, $[A]^{\psi^1}_{\psi^2}$ is independent of the choice of the representatives in ψ^1.

Proposition 10. *For any term t and assignment ψ*

$$[\![t]\!]^{\mathcal{D}}_\psi = \langle \psi^1 t, [\![t]\!]^{\mathcal{M}}_{\psi^2}\rangle$$

For any proposition A and assignment ψ

$$\llbracket A \rrbracket_\psi^{\mathcal{D}} = [A]_{\psi^2}^{\psi^1}$$

Proof. The first statement is proved by induction on the structure of t. The second by induction over the structure of A. If A is atomic, the result follows from the first statement, Definition 14 and Definition 17 and in all the other cases, from Proposition 8 and the induction hypothesis.

Proposition 11 ($\mathcal{D}$ is a model of $\equiv$). *If $A \equiv B$, then $\llbracket A \rrbracket_\psi^{\mathcal{D}} = \llbracket B \rrbracket_\psi^{\mathcal{D}}$.*

Proof. From Proposition 6, we have $[A]_{\psi^2}^{\psi^1} = [B]_{\psi^2}^{\psi^1}$, and, by Proposition 10, we get $\llbracket A \rrbracket_\psi^{\mathcal{D}} = \llbracket B \rrbracket_\psi^{\mathcal{D}}$.

Proposition 12 (Completeness of the cut free calculus). *If the sequent $\Gamma \vdash B$ is valid in the model $\mathcal{D}$, then it has a cut free proof.*

Proof. Let $\Gamma = A_1, ..., A_n$ and ϕ be an arbitrary $\mathcal{M}$-valuation. Let ψ be the $\mathcal{D}$-valuation mapping x to the pair $\langle x, \phi(x) \rangle$. If the sequent $\Gamma \vdash B$ is valid in $\mathcal{D}$, then the proposition $(A_1 \wedge ... \wedge A_n) \Rightarrow B$ is valid in $\mathcal{D}$ *i.e.* $\llbracket (A_1 \wedge ... \wedge A_n) \Rightarrow B \rrbracket_\psi^{\mathcal{D}} = \tilde{\top}$, *i.e.* $\llbracket A_1 \wedge ... \wedge A_n \rrbracket_\psi^{\mathcal{D}} \subseteq \llbracket B \rrbracket_\psi^{\mathcal{D}}$. Using Proposition 10, $[\Gamma]_{\psi^2}^{\psi^1} \subseteq [B]_{\psi^2}^{\psi^1}$. Using Proposition 9, $\psi^1 \Gamma \in [B]_{\psi^2}^{\psi^1}$, *i.e.* $\Gamma \in [B]_{\psi^2}^{\psi^1}$. Using Proposition 6, $\Gamma \vdash B$ has a cut free proof.

Theorem 1 (Cut elimination). *If the sequent $\Gamma \vdash B$ is provable, then it has a cut free proof.*

Proof. From the soundness theorem (Proposition 2) and Proposition 1, if $\Gamma \vdash B$ is provable, then it is valid in all Heyting algebra-valued models of the congruence, and in particular in the model $\mathcal{D}$. Hence, by Proposition 12, it has a cut free proof.

4 Application to simple type theory

As a particular case, we get a cut elimination proof for simple type theory.

Let us detail the model constructions in this case. Based on the language of simple type theory, we first build the truth values algebra of sequents $\mathcal{S}$ of Definition 13. Then using the super-consistency of simple type theory, we build the model $\mathcal{M}$ as in Proposition 3. In particular, we let $M_\iota = \{0\}$, $M_o = S$, and $M_{T \to U} = M_U^{M_T}$. Then, we let $D_T = \mathcal{T}_T' \times M_T$, where $\mathcal{T}_T'$ is the set of classes of terms of sort T. In particular, we have $D_\iota = \mathcal{T}_\iota' \times \{0\}$ and $D_o = \mathcal{T}_o' \times S$.

This construction is reminiscent of the definition of V-complexes, that are also ordered pairs whose first component is a term. In particular, in the definition of V-complexes of [12, 13, 1, 3, 9], we also take $\mathcal{C}_\iota = \mathcal{T}_\iota' \times \{0\}$ but $\mathcal{C}_o = \mathcal{T}_o' \times \{\textbf{false}, \textbf{true}\}$. In the intuitionistic case [3], the set $\{\textbf{false}, \textbf{true}\}$ is replaced by a complete Heyting algebra.

The difference here is that instead of using the small algebra $\{\mathbf{false}, \mathbf{true}\}$ or a Heyting algebra, we use the larger truth values algebra $\mathcal{S}$ of sequents.

Another difference is that, in our definition, we first define completely the hierarchy $\mathcal{M}$ and then perform the hybridization with the terms. Terms and functions are more intricate in the definition of V-complexes as $\mathcal{C}_{T\to U}$ is defined as a set of pairs formed with a term t of type $T \to U$ and a function f from $\mathcal{C}_T$ to $\mathcal{C}_U$ such that $f\langle t', f'\rangle$ is a pair whose first component is tt'. In our definition, in contrast, $D_{T\to U}$ is the set of pairs formed with a term t of type $T \to U$ and an element of $M_{T\to U}$, *i.e.* a function from M_T to M_U, not from D_T to D_U. When we apply a pair $\langle t, f\rangle$ of $D_{T\to U}$ to a pair $\langle t', f'\rangle$ of D_T, we just apply component-wise t to t' and f to f' and get the pair $\langle tt', ff'\rangle$. With the usual V-complexes, the result of application is the pair $f\langle t', f'\rangle$ whose first component is indeed tt', but whose second component depends on f, f', and also t'. This is indeed necessary since, in the algebra $\{\mathbf{false}, \mathbf{true}\}$ or in a Heyting algebra, $\top$ and $\top \wedge \top$ have the same interpretation and thus in the usual V-complexes models, the interpretation of $\top$ and $\top \wedge \top$ have the same second component. The only way to make the second component of $P(\top)$ and $P(\top \wedge \top)$ different (this is necessary, since they are not equivalent) is to make it depending on the first component. In our truth values algebra, in contrast, $\top$ and $\top \wedge \top$ have different interpretations. Moreover, $[\![P(\top)]\!]^{\mathcal{D}} = [\![P(\top)]\!]^{\mathcal{M}} \lhd P(\top)$ whereas $[\![P(\top \wedge \top)]\!]^{\mathcal{D}} = [\![P(\top \wedge \top)]\!]^{\mathcal{M}} \lhd P(\top \wedge \top)$. This shows that $P(\top) \in [\![P(\top)]\!]^{\mathcal{D}}$, from Definition 13 since $P(\top) \vdash P(\top)$ has a neutral cut free proof. On the same way, $P(\top \wedge \top) \in [\![P(\top \wedge \top)]\!]^{\mathcal{D}}$ for the same reason. But one does have neither $P(\top \wedge \top) \in [\![P(\top)]\!]^{\mathcal{D}}$ nor $P(\top) \in [\![P(\top \wedge \top)]\!]^{\mathcal{D}}$. Hence these truth values are distinct and incomparable.

Thus the main difference between our hybrid model construction and that of the V-complexes is that we have broken this dependency of the right component of the pair obtained by applying $\langle t, f\rangle$ to $\langle t', f'\rangle$ with respect to t' leading to a simpler construction in two steps. The reason why we have been able to do so is that starting with a larger algebra for D_o, our semantic components are more informative and thus are sufficient to define the interpretation of larger terms.

Conclusion

In this paper, we have simplified the notion of V-complex introduced for proving cut elimination for simple type theory and shown that when we use truth values algebras, this notion boils down to hybridization. Once simplified this way, it can be generalized to all super-consistent theories. This allows to relate the normalization proofs using reducibility candidates to the semantic cut elimination proofs. The latter appear to be a simplification of the former, where each proof is replaced by its conclusion. In the cut elimination proof, however, hybridization has permitted to obtain the inclusion of the interpretation of Γ in that of B, when the sequent $\Gamma \vdash B$ is provable, as a corollary of the soundness theorem. It remains to understand if such a construction can also be carried out for the normalization proof.

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